Impartial division of a dollar

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Abstract

For impartial division, each participant reports only her opinion about the fair relative shares of the other participants, and this report has no effect on her own share. If a specific division is compatible with all reports, it is implemented.

We propose a natural method meeting these requirements, for a division among four or more participants. No such method exists for a division among three participants.

1 Introduction

How to divide \$1, or any amount of a divisible commodity, in a way that respects the respective claims of the potential recipients of the money? Depending on the context, individual claims may reward contributions toward the creation of the surplus, e.g., the level of effort supplied by each "partner"; or the relative needs for the resource, as when one distribute relief after a catastrophic loss; or exogenous rights, as in a bankruptcy or inheritance situation; or a combination of these factors.

If the profile of claims is not in dispute, i.e., everyone agrees on a list of "objective" claims (e.g., verifiable liabilities in a bankruptcy), the most common ethical norm is proportionality, which we retain here[1]. But how should we proceed if the participants disagree in their evaluations of the claims, and no outside authority can enforce a compromise division?

Many reasons can lead to a lack of consensus. An objective set of claims may exist (based, perhaps, on legal rights), yet be imperfectly known to the participants (some records are lost, thus claims are not verifiable). Therefore, as in Condorcet's Jury problem, the participants will make honest mistakes when evaluating the claims. Alternatively, the claims may be inherently *subjective*, as when partners with heterogenous skills divide their joint profit. In both

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[1] It goes back to Aristotle's *Nichomachean Ethics.* A recent axiomatic literature discusses alternative norms for fair division under objective claims: see Thomson [2003] for a survey.

cases our problem is to aggregate the typically inconsistent evaluations into a reasonable division of the dollar.

We propose a division rule to reach an *impartial* compromise under any conceivable configuration of the participants' opinions. In our model, each agent sends a message related to his or her evaluation of how the dollar should be divided, and the rule "aggregates" these messages into a division of the dollar. Impartiality has two meanings here. On the one hand the *rule* itself is impartial, namely it treats all individual reports symmetrically, paying no attention to the *names* of the participants: this is the familiar principle of horizontal equity, that we call here *anonymity*. On the other hand the *messages* themselves should be impartial, in the ordinary sense that they reflect the honest opinion of each participant. Consider the following two requirements of a division rule:

- everyone reports an opinion about the (relative) shares that *other* agents deserve; no one makes any statement about *his/her own* share of the pie;

- the share of any participant is determined *exclusively* by the reports of other agents, her own report has *no influence* at all on her final share.

The first principle takes literally the old adage that a man is never a good judge of his own cause. Voicing an opinion about my own share of the dollar creates the archetypal conflict of interest, it is *prima facie* evidence of partiality. If the first principle eliminates *overt* partiality, the second one takes care of the *covert* form of partiality that begets selfish agents, tempted to use their report strategically so as to indirectly increase their share[2]. Taken together, these two properties make Borda's "honest man" an entirely credible description of our participants. They are our definition of the impartiality of the messages collected by the rule.

Our third principle conveys the idea that the eventual division should be a compromise between the various opinions. This is the *consensus* property:

- if the profile of opinions points to a *consensual* division (if there is a way to divide the pie that agrees with all individual reports), this is the solution.

Of our three requirements, only this one links the substantive content of the reports to the actual shares of the pie. It is a very weak link, because it puts no restriction at all on the outcome when there is even a modicum of disagreement among the participants. Yet, in combination with the first two properties, the consensus property has much bite.

Our main results

Our model requires at least three agents. With exactly three, there is a unique impartial and consensual division rule (Proposition 1), but that rule distributes the full dollar only in the case where the 3 reports are consistent; otherwise it distributes strictly less than \$1. By contrast, with four or more

[2] If everyone cares only about his/her share of the pie, this property is precisely the familiar *strategy-proofness* requirement.

agents, we can find many anonymous, impartial and consensual rules that always distribute the entire dollar (Theorem 1). To construct such division rules, we introduce a family of "wasteful" rules (often distributing less than the full dollar), in which the share of a given agent i is computed by aggregating the reports on the relative share of j versus i into a single ratio; Theorem 2 is a characterization of these rules based on this *separability* property. We develop our results under the simplifying assumption that all reports assign a positive share to each participant. At the cost of some restrictions on the aggregation operation mentioned above, all results are preserved when individual reports may include zero shares for some participants.

Related literature

Our model sits squarely in the literature on aggregation of individual evaluations into a collective decision, the main example of which is Condorcet Jury problem (recent contributions include List and Petitt (2002), Dokow and Holzman (2005), Nehring and Puppe (2005)). The main difference is in the nature of the decision taken by the jurors: there a public decision, here the division of a private good to which each juror is party.

The dominant formal model for non consensual division of a private good is bargaining. Our approach is orthogonal to bargaining theory, whether in its axiomatic or non cooperative form. For instance non cooperative games implementing a cooperative solution (Rubinstein (1982), Moulin (1984)) reward gaming skills and interpersonal information. Our model eliminates the strategic incentives altogether (*every* report is a dominant strategy), emulating instead the "consensual" situation where other-regarding opinions are consistent. No matter how much disagreement there is among the participants, the division rule will allocate all (or most) of the pie, so the result of disagreement bears no resemblance to the "disagreement outcome" of bargaining theory.

2 The model

In Sections 2 to 4, we assume that every participant must receive a positive share of the dollar, and that individual reports respect this constraint. When the goal is to reach a consensus, requiring that no one be entirely excluded from the distribution is a plausible constraint. At some technical cost and with some qualifications, our results extend to the case where an agent can recommend a null share for some of the other agents: Section 5.

We introduce some notation first. Write M for a set of two or more agents, M^2 for the set of pairs (i,j) in M, and $\mathcal{R}[M]$ for the following subset of $\mathbb{R}^{M^2}$:

$$r \in \mathcal{R}[M] \Leftrightarrow \{\forall i,j,k \in M,\ r_{ij} > 0,\, r_{ii} = 1,\, r_{ij} \cdot r_{jk} \cdot r_{ki} = 1\}$$

In particular, $r_{ij} \cdot r_{ji} = 1$. There is a natural bijection from $\mathcal{R}[M]$ into the interior of the $M-$simplex, namely $\overset{\circ}{\Delta}(M) = \{x \in \mathbb{R}^M | x_i > 0,\, \sum_M x_j = 1\}$. It is given by the system

$$r_{ij} = \frac{x_i}{x_j}, \text{ for all } i,j \in M \tag{1}$$

A vector $r \in \mathcal{R}[M]$ is interpreted as an evaluation of the relative shares of all agents in M: it is derived via equation (1) from a unique division x of a dollar among these agents.

Definition 1.
*Given a set N of three or more agents, a **mutual evaluation problem** is a list $(N, r^i, i \in N)$ where $r^i \in \mathcal{R}[N \diagdown \{i\}]$ for all i. This problem is **consensual** if there exists a vector $x \in \overset{\circ}{\Delta}(N)$ such that*

$$r_{ij}^k = \frac{x_i}{x_j}, \text{ for all } k \in N, \text{ and all } i, j \in N \diagdown \{k\} \tag{2}$$

The assumption $r_{ij}^k > 0$ means that in everyone's evaluation, everyone else deserves a positive share of the pie.

Lemma 1.
If $N = \{1, 2, 3\}$, the problem (N, r) is consensual if and only if $r_{23}^1 \cdot r_{31}^2 \cdot r_{12}^3 = 1$. If $|N| \geq 4$, the problem (N, r) is consensual if and only if

$$r_{ij}^k = r_{ij}^l \text{ for all } k, l \in N, \text{and all } k, l \in N \diagdown \{i, j\}$$

If (N, r) is consensual, the corresponding division is

$$x_i = \frac{1}{1 + \sum_{N \diagdown \{i\}} r_{ji}}$$

where r_{ji} is the common value of r_{ij}^k (the naturalness of this expression can be seen by replacing r_{ji} by $\frac{x_j}{x_i}$).

We omit the straightforward proof.

Definition 2.
*Given N, a **division rule** f assigns to each mutual evaluation problem (N, r) a vector $f(N, r) = x \in \mathbb{R}_+^N$. The rule f is **exact** if $\sum_N x_i = 1$ for all r; it is **(gain-)feasible** if $\sum_N x_i \leq 1$ for all r; it is **cost-feasible** if $\sum_N x_i \geq 1$ for all r*

The dollar to be divided can be interpreted as a net *gain,* or as a net *cost.* If the division rule is exact, both interpretations are valid. For an inexact rule, the feasibility constraint depends on the interpretation of the dollar as a gain or a cost. Although our main interest is in exact rules, gain-feasible rules are also relevant (Propositions 1 and 2): they are a key ingredient in the construction of the exact rules of Theorem 1. On the other hand, cost feasible rules play no role in our model (as explained in Propositions 1 and 2), so we focus on the gain interpretation: when we speak below of a feasible rule, we always mean gain-feasible.

Definition 3.
*Given N, the division rule f is **consensual** if it chooses the consensus division when the problem (N, r) is consensual. It is **anonymous** if it is a symmetric*

mapping with respect to permutations of N. It is ***impartial*** *if the share of an agent is independent of her own report: for all r, r' and all i*

$$\{r^j = r'^j \text{ for all } j \in N \diagdown \{i\}\} \Rightarrow f_i(N, r) = f_i(N, r')$$

If I care only about the size of my own share, impartiality is the familiar strategy-proofness property: my report is *selfless*, it has no impact on my own welfare, therefore "gaming" is irrelevant. This is no longer true if I care about the profile of shares of the other agents, or if a coalition of agents choose their reports strategically.

It is easy to construct an anonymous, impartial and exact division rule. Given a problem (N, r), let $x^i \in \overset{\circ}{\Delta}(N \diagdown \{i\})$ be the division of the dollar among $N \diagdown \{i\}$ proposed by agent i (system (1)), and $x^i_* \in \overset{\circ}{\Delta}(N)$ be the "same" division of the dollar among N where the share of i is zero: then $f(r) = \frac{1}{n}\sum_N x^i_*$ is such a rule. But this rule is not consensual.

Proposition 1.
For $N = \{1, 2, 3\}$, there is a unique impartial and consensual division rule f^*:

$$f^*(r) = (\frac{1}{1 + r^2_{31} + r^3_{21}}, \frac{1}{1 + r^3_{12} + r^1_{32}}, \frac{1}{1 + r^1_{23} + r^2_{13}}) \text{ for all } r \tag{3}$$

This rule is anonymous and feasible; it distributes the whole dollar only if the problem is consensual:

$$\begin{aligned} r^1_{23} \cdot r^2_{31} \cdot r^3_{12} &\neq 1 \Rightarrow \sum_1^3 f^*_i(r) < 1 \\ r^1_{23} \cdot r^2_{31} \cdot r^3_{12} &= 1 \Rightarrow \sum_1^3 f^*_i(r) = 1 \end{aligned}$$

Proof.
Pick a rule f impartial and consensual. The share of agent 1 takes the form $f_1(r^2, r^3)$. For any r^2, r^3, choose r^1 so that (r^1, r^2, r^3) is consensual: $r^1_{23} = r^3_{21} r^2_{13}$, and $r^1_{23} r^1_{32} = 1$. The solution x of system (2) is then

$$x = (\frac{1}{1 + r^2_{31} + r^3_{21}}, \frac{1}{r^3_{12}(1 + r^2_{31} + r^3_{21})}, \frac{1}{r^2_{13}(1 + r^2_{31} + r^3_{21})})$$

and consensuality gives the desired share $f^*_1(r)$. Therefore $f = f^*$. Feasibility of f^*, and the fact that f^* wastes some money whenever r is not consensual, follow at once from Lemma 4 in the Appendix. ■

A consequence of Proposition 1 is that among three agents, no impartial and consensual division rule is cost-feasible.

The above impossibility result no longer holds with four or more agents: we construct in Section 4 impartial, consensual and exact division rules for such problems. Our construction starts by a family of feasible yet inexact rules generalizing f^* in (3) to any number of agents.

3 Separable division rules

Consider agent 1. For each $j, j \neq 1$, every agent i in $N \diagdown \{1, j\}$ contributes an opinion r^i_{j1} about the relative shares of j and 1. The rules we construct aggregate the ratios $\{r^i_{j1}, i \in N \diagdown \{1, j\}\}$ into a single representative ratio.

Definition 4.
*Given an integer m, an m-***aggregator** *is a symmetric, continuous and non decreasing mapping ρ^m from $\mathbb{R}^m_{++}$ into $\mathbb{R}_{++}$, such that*

$$\rho^m(a, \cdots, a) = a \text{ for all } a > 0$$

The familiar arithmetic, geometric, and harmonic means are examples of aggregators. Of particular interest to us are the "rank order" aggregators and their convex combinations. For all $z \in \mathbb{R}^m_{++}$, let z^* obtain from z by rearranging its coordinates increasingly: $z^*_1 \leq z^*_2 \leq \cdots \leq z^*_m$ For any set of convex weights $\lambda \in \Delta(m)$, the equation $\rho^m(z) = \sum_1^m \lambda_i z^*_i$ defines an aggregator. Note that for *any* aggregator ρ^m we have $\min_i z_i \leq \rho^m(z) \leq \max_i z_i$, therefore $\min_i z_i$ and $\max_i z_i$ are respectively the smallest and largest aggregators.

Proposition 2.
Fix $N = \{1, 2, \cdots, n\}, n \geq 3$, and a $(n-2)$-aggregator ρ. For any problem (N, r), write $\rho(r_{ji}) = \rho(r^k_{ji}; k \in N \diagdown \{i, j\})$, and define the division rule f^ρ as follows

$$f^\rho_i(r) = \frac{1}{1 + \sum_{j \in N \diagdown \{i\}} \rho(r_{ji})} \text{ for all } i \text{ and } r \tag{4}$$

i) This rule is anonymous, impartial and consensual;
ii) It is not cost-feasible:

$$\inf_r \sum_1^n f^\rho_i(r) = 0$$

iii) It is feasible if and only if

$$\rho(z) \cdot \rho(\frac{1}{z}) \geq 1 \text{ for all } z \in \mathbb{R}^{n-2}_{++}, \text{ where } \frac{1}{z} = (\frac{1}{z_1}, \cdots, \frac{1}{z_{n-2}}) \tag{5}$$

iv) If, in addition to (5), ρ satisfies for all z

$$\rho(z) \cdot \rho(\frac{1}{z}) = 1 \Longleftrightarrow z_1 = z_2 = \cdots = z_{n-2} \tag{6}$$

then for all problem r we have

$$\sum_1^n f^\rho_i(r) = 1 \Longleftrightarrow r \text{ is consensual.} \tag{7}$$

Proof.
Statement i) is clear. *For statement ii)*, fix $\varepsilon > 0$, write θ for the bijection of $\{1, \cdots, n\}$ into itself

$$\theta(i) = i + 1 \text{ for } i < n, \theta(n) = 1$$

and check that the following equations define a unique problem r

$$r^j_{i,\theta(i)} = \varepsilon, \text{ for all } i, j \text{ such that } j \neq i, \theta(i).$$

The unanimity property of ρ implies $\rho(r_{\theta(i),i}) = \frac{1}{\varepsilon}$ for all i, therefore $f^\rho_i(r) \leq \frac{1}{1+\rho(r_{\theta(i),i})} \leq \varepsilon$. Choosing ε arbitrarily small establishes the claim.
For statement iii), assume first that ρ meets inequalities (5). Fix a problem (N, r), then apply the first statement of Lemma 4 to $y_{ij} = \rho(r_{ij})$, taking into account $r^k_{ij} r^k_{ji} = 1$: this gives $\sum_N f^\rho_i(r) \leq 1$. Conversely we must show that ρ satisfies (5) if f^ρ is feasible. Fix $z_1, \cdots, z_{n-2} \in \mathbb{R}^{n-2}_{++}$ and consider the following profile of reports:

$$\begin{aligned} r^1_{2i} &= \lambda, r^1_{ij} = 1 \text{ for all } i,j \geq 3;\ r^2_{1i} = \lambda, r^2_{ij} = 1 \text{ for all } i,j \geq 3 \\ \text{for } k &\geq 3:\ r^k_{12} = z_k, r^k_{1i} = \lambda z_k, r^k_{2i} = \lambda, r^k_{ij} = 1 \text{ for all } i,j \geq 3 \end{aligned}$$

and the other coordinates deduced by $r^k_{ij} r^k_{ji} = 1$. Check that for all $i \geq 3$, $f^\rho_i(r) \to 0$ as $\lambda \to +\infty$, whereas

$$f^\rho_1(r) \to \frac{1}{1 + \rho(\frac{1}{z_1}, \cdots, \frac{1}{z_{n-2}})};\ f^\rho_2(r) \to \frac{1}{1 + \rho(z_1, \cdots, z_{n-2})}$$

Now feasibility implies inequality (5) at once.
For statement iv) we apply again the first statement of Lemma 4: for any r, equality $\sum_1^n f^\rho_i(r) = 1$ holds only if $\rho(r_{ij}) \cdot \rho(r_{ji}) = 1$ for all i, j. If ρ meets (6) this implies that r^k_{ij} is independent of $k \in N \diagdown \{i, j\}$, thus r is consensual by Lemma 1. ■

For $n = 3$, there is only one aggregator ρ^{n-2}, and Proposition 2 repeats Proposition 1.

For $n \geq 4$, the *harmonic* mean fails (5)[3], therefore this aggregator is not useful in our problem.

The *geometric* mean ρ_g meets (5) but fails (6); in fact $\rho_g(z) \cdot \rho_g(\frac{1}{z}) = 1$ holds for *every* $z \in \mathbb{R}^{n-2}_{++}$.

The *arithmetic* mean ρ_a satisfies (5) and (6) (a simple consequence of the Schwartz inequality; or see Lemma 2 below). Therefore the corresponding rule divides the entire dollar *only* in a consensual problem.

Recall that $\rho_g \leq \rho_a$. By (4) this implies $f^{\rho_a} \leq f^{\rho_g}$, in particular the rule f^{ρ_g} is less "wasteful" than f^{ρ_a}; for instance f^{ρ_g} allocates the entire dollar in many non consensual problems. Thus the desirability of property (7) is not straightforward. If the participants engage in extensive discussions before sending their reports, it gives them a strong collective incentive to reach a consensus. If on the other hand individual reports result from decentralized introspection, we

[3] This is clear for $n - 2 = 2$, and just as easy to show for any n.

will prefer a rule that wastes as little money as rarely as possible, and to this end a rule ensuring $\rho(z) \cdot \rho(\frac{1}{z}) = 1$ for all z is clearly optimal: no other aggregator ρ' can meet (5) and be everywhere smaller than ρ. Examples include the geometric mean, and, if $m = 2m' - 1$ is odd, the median aggregator $\rho(z) = z^*_{m'}$.

Proposition 4 in section 5 gives an alternative justification of (7) for the aggregators used in Theorem 1 to construct exact rules: when this property holds, it is especially difficult to assign a zero share to any agent.

Lemma 2.

Consider a convex combination of rank order aggregators $\rho(z) = \sum_1^{n-2} \lambda_i z^*_i$, *and write* $m = \lfloor a \rfloor$ *for the largest integer no greater than* a.

i) Property (5) holds if and only if

$$\sum_1^k \lambda_i \leq \sum_1^k \lambda_{n-i-1}, \text{ for all } k = 1, \cdots, \lfloor \frac{n}{2} \rfloor - 1 \tag{8}$$

ii) Given (8), property (6) holds if and only if $\lambda_{n-2} > 0$.

Proof.

Statement i). For any $\lambda, \lambda' \in \Delta(n-2)$, we define the familiar stochastic dominance relation:

$$\lambda \succeq \lambda' \Leftrightarrow \sum_1^k \lambda_i \leq \sum_1^k \lambda'_i \text{ for } k = 1, \cdots, n-3$$

Call λ *symmetric* if $\lambda_i = \lambda_{n-i-1}$ for all $i = 1, \cdots, \lfloor \frac{n}{2} \rfloor$. We claim that λ satisfies system (8) if and only if

$$\lambda \succeq \lambda' \text{ for some symmetric } \lambda'$$

For *if*, note that $\lambda \succeq \lambda'$ implies $\lambda_1 \leq \lambda'_1$ and $\lambda'_{n-2} \leq \lambda_{n-2}$, hence $\lambda_1 \leq \lambda_{n-2}$ by the symmetry of λ'. Similarly $\lambda_1 + \lambda_2 \leq \lambda'_1 + \lambda'_2 = \lambda'_{n-3} + \lambda'_{n-2} \leq \lambda_{n-3} + \lambda_{n-2}$, and so on. For *only if*, assume λ satisfies system (8), define λ', symmetric, by $\lambda'_i = \lambda'_{n-i-1} = \frac{\lambda_i + \lambda_{n-i-1}}{2}$, and check $\lambda \succeq \lambda'$. This proves the claim.

Assume now that λ is symmetric and compute for all z

$$\rho(\frac{1}{z}) = \sum_1^{n-2} \frac{\lambda_i}{z^*_{n-i-1}} = \sum_1^{n-2} \frac{\lambda_i}{z^*_i} \Rightarrow \rho(z) \cdot \rho(\frac{1}{z}) = (\sum_1^{n-2} \lambda_i z^*_i) \cdot (\sum_1^{n-2} \frac{\lambda_i}{z^*_i}) \geq \sum_1^{n-2} \lambda_i = 1$$

where the last inequality follows by applying Schwarz's inequality to the vectors $(\sqrt[2]{\lambda_i z^*_i})$ and $(\sqrt[2]{\frac{\lambda_i}{z^*_i}})$. Thus (5) holds for symmetric λ. Next assume $\lambda \succeq \lambda'$, write $a \cdot b$ for the scalar product in $\mathbb{R}^{n-2}$, and note that $\rho_\lambda(z) = \lambda \cdot z^* \geq \lambda' \cdot z^* = \rho_{\lambda'}(z)$, as z^*_i is non decreasing in i. Therefore ρ_λ meets (5) if $\rho_{\lambda'}$ does. The *if* statement is proven.

For *only if*, fix $k, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$, $\varepsilon > 0$, and define z by $z_i = 1 - \varepsilon$ for $i \leq k$,$= 1$ for $k+1 \leq i \leq n-k-2$,$= 1+\varepsilon$ for $n-k-1 \leq i \leq n-2$. Fix λ of which the corresponding aggregator meets (5), and set $a = \sum_1^k \lambda_i, b = \sum_{n-k-1}^{n-2} \lambda_i, c = 1 - a - b$. Compute

$$\rho(z) \cdot \rho(\frac{1}{z}) = ((1-\varepsilon)a + c + (1+\varepsilon)b) \cdot (\frac{a}{1+\varepsilon} + c + \frac{b}{1-\varepsilon})$$

The RHS equals 1 for $\varepsilon = 0$, hence by (5) its derivative at $\varepsilon = 0$ is non negative. This implies $a \leq b$ as desired.
Statement ii). Take λ satisfying (8), so it dominates a symmetric distribution λ'. If $\lambda_{n-2} > 0$, we can choose λ' such that $\lambda'_1 = \lambda'_{n-2} > 0$. By the above proof, $\rho_\lambda(z) \cdot \rho_\lambda(\frac{1}{z}) = 1$ implies $\rho_{\lambda'}(z) \cdot \rho_{\lambda'}(\frac{1}{z}) = 1$, and by Schwarz's inequality, the vectors $(\sqrt[2]{\lambda'_i z^*_i})$ and $(\sqrt[2]{\frac{\lambda'_i}{z^*_i}})$ must be parallel. This gives at once $z^*_1 = z^*_{n-2}$, namely $z_1 = z_2 = \cdots = z_{n-2}$. This proves *if*. Conversely, if $\lambda_{n-2} = 0$, (8) implies $\lambda_1 = 0$, so $z = (1, 2, \cdots, 2, 3)$ has $\rho(z) \cdot \rho(\frac{1}{z}) = 1$, yet all coordinates of z are not equal. ■

The proof shows that (8) holds if the weights λ_i are symmetric. Examples include the arithmetic mean, the median $\lambda_{\lfloor \frac{n}{2} \rfloor} = 1$ if n is odd, $\lambda_{\lfloor \frac{n}{2} \rfloor - 1} = \lambda_{\lfloor \frac{n}{2} \rfloor} = \frac{1}{2}$ if n is even, and $\rho(z) = \frac{1}{2}(\min_i z_i + \max_i z_i)$. Another sufficient condition for (8) is that the support of λ be entirely contained in $\{\lfloor \frac{n}{2} \rfloor, \cdots, n-2\}$: examples are the rank order aggregators $\rho(z) = z^*_k$ for $\lfloor \frac{n}{2} \rfloor \leq k \leq n-2$.

We conclude this section by explaining its title, namely the separability property shared by all division rules, not necessarily feasible, of the form (4) for some aggregator ρ. Consider two profiles of reports r^{-1} and s^{-1} by the agents other than agent 1, that only differ in the reports r^i_{21}, for $i = 3, \cdots, n$. As $\rho(r_{j1}) = \rho(s_{j1})$ for all $j = 3, \cdots, n$, we can tell if agent 1's share goes up or down simply by comparing $\rho(r_{21})$ and $\rho(s_{21})$, and for this we only need to know the numbers r^i_{21} and s^i_{21}, for $i = 3, \cdots, n$. Thus the impact of a change in the reports r^i_{21} on agent 1's share, can be evaluated independently of the rest of the reports. In the Appendix we show that, in combination with anonymity, impartiality and consensuality, this property essentially characterizes the division rules (4).

4 Four agents or more: exact rules

Given N, with $|N| = n$, we choose *two* aggregators of different dimensions, ρ^{n-2} and ρ^{n-3}. We use the notation in Proposition 2 as well as $\rho^{n-3}(r^{-j}_{ki}) = \rho^{n-3}(r^l_{ki}; l \in N \diagdown \{i, j, k\})$, and define a division rule as follows:

$$f_i(r) = \frac{1}{n}[1 + \sum_{j \in N \diagdown \{i\}} (f^{-j}_i(r) - f^{-i}_j(r))] \text{ where} \tag{9}$$

$$f^{-j}_i(r) = \frac{1}{1 + \rho^{n-2}(r_{ji}) + \sum_{k \in N \diagdown \{i,j\}} \rho^{n-3}(r^{-j}_{ki})} \text{ for all } i, j \text{ distinct} \tag{10}$$

Theorem 1.

Fix N, s.t. $|N| = n \geq 4$. If both aggregators ρ^{n-2} and ρ^{n-3} meet inequalities (5), equations (9) and (10) define an anonymous, impartial, consensual and exact division rule.

Proof.

Think of agent 1 as the residual claimant, and distribute to agent $j, j = 2, \cdots, n$ a share resembling her share under $f^{\rho^{n-2}}$, with the difference that we omit agent 1's report in equation (4): this share is $f_j^{-1}(r)$, where the term $\rho(r_{kj})$ in (4) has become $\rho^{n-3}(r_{ki}^{-j})$ because we ignore 1's report, and the term $\rho(r_{1j})$ in (4) is unchanged because it does not depend on r^1 anyway. We claim that agent 1's residual share $1 - \sum_2^{n-2} f_j^{-1}(r)$ is non negative. Setting $N^* = N \diagdown \{1\}$ we have

$$f_j^{-1}(r) = \frac{1}{1 + \rho^{n-2}(r_{1j}) + \sum_{k \in N \diagdown \{1,j\}} \rho^{n-3}(r_{kj}^{-1})} \leq \frac{1}{1 + \sum_{k \in N^* \diagdown \{j\}} \rho^{n-3}(r_{kj}^{-1})}$$

where the RHS is simply the share of agent j in the division rule among N^* corresponding to the aggregator ρ^{n-3} in equation (4). Thus Proposition 2 implies $\sum_{j \in N^*} f_j^{-1}(r) \leq 1$ as claimed.

We just constructed an exact yet non anonymous division rule, in which agent 1 is the passive residual claimant. This rule is obviously impartial. To check that it is consensual, suppose r is associated with the consensual division x. Then for $j \geq 2$ we have $\rho^{n-2}(r_{1j}) = \frac{x_1}{x_j}, \rho^{n-3}(r_{kj}^{-1}) = \frac{x_k}{x_j}$ so that $f_j^{-1}(r) = x_j$, and 1's residual share is $1 - \sum_2^{n-2} x_j = x_1$.

The rule f defined by (9)(10) is simply the average of the n asymmetric rules where each agent in turn is the residual claimant. The latter is anonymous, and the other three properties are preserved by convex combinations. ■

We emphasize that there are many other rules meeting the four properties in Theorem 1. First these properties are stable by convex combinations, and we have many choices of the aggregators ρ^{n-2} *and* ρ^{n-3}. Next we can construct rules that do not need any aggregator. Suppose $n = 4$; when 1 is the residual claimant, the share $\widetilde{f_2^{-1}}(r^{-1})$ is now an average of the two terms $\frac{1}{1+r_{12}^i + r_{32}^4 + r_{42}^3}$ for $i = 3, 4$, and other shares $\widetilde{f_j^{-i}}(r^{-i})$ are deduced by symmetry. Now $f_1(r)$ takes the form $f_1(r) = \frac{1}{4}[1 + \frac{1}{2}T - \frac{1}{2}T']$, where T is the sum of 6 terms like $\frac{1}{1+r_{21}^3 + r_{31}^4 + r_{41}^3}$, and T' that of 6 terms like $\frac{1}{1+r_{12}^3 + r_{32}^4 + r_{42}^3}$.

We conclude Section 4 by listing several more desirable features shared by all rules in Theorem 1, irrespective of the choice of the two aggregators.

First a simple **monotonicity** property: if agent k alone changes her report in favor of agent i, keeping the ratios between all agents in $N \diagdown \{i, k\}$ unchanged, the share of agent i cannot decrease. This is clear from the monotonicity of ρ: in equation (4) all terms $\rho(r_{ji}), j \neq k, i$ go down or stay put, all terms $\rho(r_{ij})$ go up or stay put, and all terms $\rho(r_{jj'}), j, j' \neq i$ are unchanged.

Next we have two **continuity** properties. The share of every agent depends continuously upon the profile of individual reports (because aggregators are continuous functions). Moreover, if the reports are *almost* consensual, then the outcome is similarly close to the almost-consensus. Define the problem (N, r) to be **ε-consensual** if there exists a vector $x \in \Delta(N)$ such that

$$r_{ij}^k \leq (1+\varepsilon)\frac{x_i}{x_j}, \text{ for all } k \in N, \text{ and all } i,j \in N\diagdown\{k\}$$

If the problem (N, r) is ε-consensual with respect to x, then $|f(N,r)-x| = O(\varepsilon)$. In words, if all opinions are "close" to an underlying compromise x, our methods implement a division which is comparably close.

Finally we note that the (inexact) rule $f^{\max}$, namely the smallest of all separable rules in Proposition 2, is a lower bound for all the rules constructed in Theorem 1:

$$f_i(r) \geq f_i^{\max}(r) = \frac{1}{1+\sum_{j\in N\diagdown\{i\}} \max_{N\diagdown\{i,j\}} r_{ji}^k} \tag{11}$$

Lemma 3.
Every division rule in Theorem 1 meets inequalities (11) for all i and r.

Proof.
We fix i and r, and we refine the inequality $\sum_{j\in N^*} f_j^{-i}(r) \leq 1$ with the help of Lemma 4 in section 6.1. Set $a_j = \rho^{n-2}(r_{ij})$, and $y_{kj} = \rho^{n-3}(r_{kj}^{-i})$ for $k, j \in N^*$. Inequalities (5) for ρ^{n-3} imply $y_{kj}y_{jk} \geq 1$, so we apply the second statement of Lemma 4 to N^*, together with property (5) for ρ^{n-2}:

$$\sum_{j\in N^*} f_j^{-i}(r) = \sum_{j\in N^*} \frac{1}{1+a_j+\sum_{N^*\diagdown\{j\}} y_{kj}} \leq 1-\frac{1}{1+\sum_{N^*}\frac{1}{a_j}} \leq 1-\frac{1}{1+\sum_{N^*}\rho^{n-2}(r_{ji})}$$

As $\rho^{n-2}(r_{ji}) \leq \max_{N\diagdown\{i,j\}} r_{ji}^k$, we now have

$$1-\sum_{j\in N^*} f_j^{-i}(r) \geq \frac{1}{1+\sum_{N^*}\rho^{n-2}(r_{ji})} \geq f_i^{\max}(r)$$

Finally, $\rho^{n-3}(r_{ki}^{-j}) = \rho^{n-3}(r_{ki}^l; l \in N\diagdown\{i,j,k\}) \leq \max_{N\diagdown\{k,i\}} r_{ji}^l$, therefore for all $j \in N^*$

$$f_i^{-j}(r) \geq \frac{1}{1+\sum_{k\in N^*}\max_{N\diagdown\{k,i\}} r_{ji}^l} = f_i^{\max}(r)$$

Combining the last two inequalities in (6), the desired conclusion (9) obtains.■

5 Zero shares

Reporting that some agents deserve no share of the dollar is ruled out by the definition of evaluations r^i used so far (Definition 1), and indeed the separable rules in Section 3, and their exact extension in Section 4, guarantee a positive share to every participant. We now enlarge the domain $\mathcal{R}[M]$ of individual reports r^i to allow for zero shares: the new set is denoted $\mathcal{R}^*[M]$.

A report $r \in \mathcal{R}^*[M]$ consists of a pair $(S, r[S])$, where $S \subsetneq M$ and $r[S] \in \mathcal{R}[S]$. The interpretation is that agents in S receive a positive share and the others get zero. There is a natural bijection from $\mathcal{R}^*[M]$ into the $M-$simplex $\Delta(M) = \{x \in \mathbb{R}^M | x_i \geq 0, \sum_M x_j = 1\}$.

Definition 5.
Given N, a mutual evaluation problem is a list $(N, r^i, i \in N)$ where $r^i \in \mathcal{R}^*[N \diagdown \{i\}]$ for all i. This problem is ***consensual*** *if*

- *either there exists $N_+, N_+ \subseteq N$ and $|N_+| \geq 2$, such that $S^i = N_+ \diagdown \{i\}$ for all i, and there exists a vector $x \in \Delta(N_+)$ such that $r^k_{ij} = \frac{x_i}{x_j}$, for all $k \in N$, and all $i, j \in N_+ \diagdown \{k\}$,*
- *or there exists $i_+ \in N$ such that $S^i = \{i_+\}$ for all $i \in N \diagdown \{i_+\}$.*

The definition of a division rule (Definition 2) is unchanged, and so is that of an anonymous or impartial rule. In a consensual problem with $|N_+| \geq 2$, a consensual rule must divide the dollar as the corresponding $x \in \Delta(N_+)$; if on the other hand $S^i = \{i_+\}$ for all $i \neq i_+$, consensuality requires to give everything to i_+ (in the latter case agent i_+'s own opinion is irrelevant).

For three-person problems, Proposition 1 is preserved, including equation (3), provided we adopt the convention that if agent i reports $S^i = \{j\}$, then $r^i_{jk} = \infty, r^i_{kj} = 0$. As in the proof of Proposition 1 we compute $f_1(r^2, r^3) = f_1(r^2_{31}, r^3_{21})$ for a consensual rule. If $r^2_{31} = \infty$ then $S^2 = \{3\}$ so a report $r^1_{23} = 0 \Leftrightarrow S^1 = \{3\}$ makes (r^1, r^2, r^3) consensual (Definition 5), implying $f_1(r^2, r^3) = 0$. Thus the share of agent 1 is zero if at least one of agents $2, 3$ gives him nothing. If $r^2_{31} = r^3_{21} = 0$, we have $S^2 = S^3 = \{1\}$, so (r^1, r^2, r^3) is consensual for any r^1 and $f_1(r^2, r^3) = 1$. If $r^2_{31} = 0$ and $0 < r^3_{21} < \infty$, then a report $r^1_{23} = \infty \Leftrightarrow S^1 = \{2\}$ makes (r^1, r^2, r^3) consensual with $N_+ = \{1, 2\}$ and $(x_1, x_2) = (\frac{1}{1+r^3_{21}}, \frac{r^3_{21}}{1+r^3_{21}})$.

Next we extend the separable rules of Section 3 to allow for zero shares. Equation (4) now involves ratios r^k_{ji} that can be 0 or ∞, and aggregators ρ^m defined on $[0, \infty]^m$. Specifically we define $r^k_{ji} = 0$ if $i \in S^k, j \notin S^k$, and $r^k_{ji} = \infty$ if $i \notin S^k, j \in S^k$. There is no natural definition of r^k_{ji} if $j, i \notin S^k$, and positing an arbitrary value for r^k_{ji} in this case may change the value of the term $\rho(r_{ji})$, hence of the share f^ρ_i in (4). Fortunately, some choices of the aggregator ρ remove this difficulty. If $j, i \notin S^k$, r^k_{ji} is not defined, and neither is $\rho(r_{ji})$; however $r^k_{j'i} = \infty$ for any j' in S^k and for some aggregators this implies $\rho(r_{j'i}) = \infty$, so that $\sum_{l \in N \diagdown \{i\}} \rho(r_{li}) = \infty$ and (4) reads $f^\rho_i = 0$ irrespective of r^k_{ji}. For this argument the key property of ρ is

$$\text{for all } z \in [0, \infty]^{n-2} \ \max_i z_i = \infty \Rightarrow \rho(z) = \infty \tag{12}$$

In order to state the analog of Proposition 2, we endow $[0, \infty]$ with the standard topology, and define an (extended) m-aggregator to be a symmetric, continuous and non decreasing mapping ρ^m from $[0, \infty]^m$ into $[0, \infty]$ such that

$\rho^m(a, \cdots, a) = a$ for all $a \in [0, \infty]$. The operation $z \to \frac{1}{z}$ extends continuously to $[0, \infty]^m$; the multiplication in $[0, \infty]$ extends as well, with the convention $0 \cdot \infty = 1$. However this extension is not continuous.

Properties (5) and (6) are now meaningful for extended aggregators. Using the notation in Proposition 2, we have

Proposition 3.
Fix $N = \{1, 2, \cdots, n\}, n \geq 3$, and a $(n-2)$-aggregator ρ on $[0, \infty]^{n-2}$ satisfying (12).
i) The division rule f^ρ given by (4) is well defined, anonymous, impartial and consensual;
ii) It is not cost-feasible;
iii) It is feasible if and only if ρ satisfies (5);
iv) If ρ satisfies (5), (6), as well as

$$\text{for all } z \in [0, \infty]^{n-2}, \rho(z) = 0 \Rightarrow z = 0 \tag{13}$$

then $f^\rho(r)$ divides the entire dollar if and only if the problem r is consensual.

Proof.
We explained before the statement of Proposition 3 why property (12) ensures that equation (4) is well defined. Anonymity and impartiality are clear. For consensuality, note that if $|N_+| \geq 2$, $f_i^\rho(r) = 0$ for all $i \notin N_+$, and the sum $\sum f_i^\rho(r)$ reduces to that for a consensual problem in N_+ with all reports $0 < r_{jk}^i < \infty$.

Statement *ii)* requires no proof either: we use the same profile of reports as in the proof of Proposition 2.

The *if* part of statement *iii)* follows immediately from Lemma 5 extending Lemma 4 in the Appendix. For the *only if* part, we fix $z_1, \cdots, z_{n-2} \in [0, \infty]^{n-2}$ and observe that if $z_k = \infty$ for some k, (12) implies $\rho(z) = \infty$ hence $\rho(z) \cdot \rho(\frac{1}{z}) \geq 1$ by our convention $0 \cdot \infty = 1$. Similarly $z_k = 0$ for some k implies $\rho(\frac{1}{z}) = \infty$. Thus we are left with the case $z_1, \cdots, z_{n-2} \in \mathbb{R}_{++}^{n-2}$ as in Proposition 2.

Only statement *iv)* requires a little work. Fix a problem r such that $\sum_N f_i^\rho(r) = 1$. Lemma 5 implies the existence of a non empty subset N_+ such that, if we set $N_- = N \diagdown N_+$

$$\begin{aligned} \rho(r_{ij}) &= 0 \text{ if } i \in N_-, j \in N_+; = \infty \text{ if } i \in N_+, j \in N_- \\ \text{and if } |N_+| &\geq 2, \ (\rho(r_{ij}))_{i,j \in N_+} \in \mathcal{R}[N_+] \end{aligned}$$

If $N_- = \varnothing$ we are back to Proposition 2, so we assume $N_- \neq \varnothing$ and distinguish two cases. If $N_+ = \{i_+\}$ we have $\rho(r_{ii_+}) = 0$ for all $i \neq i_+$ therefore by (13) $r_{ii_+}^k = 0$ for all $i, k \neq i_+$, so $S^k = \{i_+\}$ for all $k \neq i_+$ and our problem is consensual (and of course (4) implies that f^ρ gives everything to i_+). The second case is $|N_+| \geq 2$. Then $\rho(r_{ij}) = 0$ for all $i \in N_-, j \in N_+$, and by (13) $r_{ij}^k = 0$ for all $k = i, j$. This implies $S^k \subseteq N_+$ for all $k \in N$. From $(\rho(r_{ij})) \in \mathcal{R}[N_+]$ we get $0 < \rho(r_{ij}) < \infty$ for all $i, j \in N_+$, so by (12) $r_{ij}^k < \infty$ for all such i, j and $k \neq i, j$. This gives $S^k = N_+$ for all $k \in N$. Now for $i, j \in N_+$,

$\rho(r_{ij}) \cdot \rho(r_{ji}) = 1$ implies by (6) that r_{ij}^k does not depend on k, and we conclude that the restriction of r to N_+ is consensual. ■

Recall that the extended report $r^i \in \mathcal{R}^*[N \diagdown \{i\}]$ can be described as a point in the simplex $\Delta(N \diagdown \{i\})$; if we endow $\mathcal{R}^*[N \diagdown \{i\}]$ with the topology induced by that of $\Delta(N \diagdown \{i\})$, one checks that the rule f^ρ is continuous (provided ρ satisfies (12)).

For the convex combinations of rank order aggregators $\rho(z) = \sum_1^{n-2} \lambda_i z_i^*$, both (12) and (13) follow at once from $\lambda_{n-2} > 0$. From Lemma 2 we see that the inequalities (8) and $\lambda_{n-2} > 0$ ensure that f^ρ meets all the properties listed in Proposition 3. This leaves us with a large set of aggregators to choose from.

Extending the definition of the exact rules in Section 4 is now a simple matter. We choose two aggregators ρ^{n-2} and ρ^{n-3} meeting (5) and (12), and define $f_i^{-j}(r)$ by the same equation (10). This definition is meaningful for the same reason as that of f^ρ is. The statement of Theorem 1 is now identical in the extended model, except for the additional assumption (12) on ρ^{n-2} and ρ^{n-3}.

The new feature of the extended division rules, whether exact or not, is that an agent can receive no money at all. For instance consider the rule f^ρ in Proposition 3 where the $(n-2)$-aggregator ρ is a convex combination of rank orders such that $\lambda_{n-2} > 0$. If agent k reports $i \notin S^k \Leftrightarrow \{r_{ji}^k = \infty$ for all $j \in S^k\}$, then $f_i^\rho(r) = 0$: in other words any agent can single-handedly bar any other agent (even $n-2$ other agents) from any positive benefit! What we would like instead is to protect each participant from the ill-will of a single "enemy".

Within the family of exact rules in Theorem 1 it turns out that the combination of properties (6) and (13) implies maximal protection for each participant in the following sense: agent i will receive no money at all only if *every* other participant agrees that this is fair, and moreover they agree on the relative shares of the dollar among themselves. And a symmetrical statement for the case where agent i receives the whole dollar (resp., is assigned the entire cost): this will only happen if all other agents agree that i should receive (resp., pay) the entire dollar (resp., cost).

Contrast the above "protection" with the situation when ρ^{n-2} and ρ^{n-3} are for instance the median aggregators: if a strict majority of $N \diagdown \{i\}$ reports that i should get a zero share, and the shares $f_j^{-i}(r), j \in N \diagdown \{i\}$ sum to 1 (which does not require consensus over the relative shares in $N \diagdown \{i\}$), then agent i gets zero.

Given N, i and a report $r^j \in \mathcal{R}^*[N \diagdown \{j\}]$ for some $j \neq i$, note that if $i \notin S^j$ the projection $r^j[-i]$ of r^j on $N \diagdown \{i\}$ (an element of $\mathcal{R}^*[(N \diagdown \{i\}) \diagdown \{j\}]$) is well defined because r^j assigns a zero share to i.

Proposition 4.
Fix $N = \{1, 2, \cdots, n\}, n \geq 4$, and two aggregators, ρ^{n-2} on $[0, \infty]^{n-2}$ and ρ^{n-3} on $[0, \infty]^{n-3}$, satisfying (5),(6),(12),and (13). Let f be the exact division rule defined by equations (9) and (10). For all problem r and all agent i, we have:

$$f_i(r) = 0 \Longleftrightarrow \{i \notin S^j \text{ for all } j \in N \diagdown \{i\} \text{ and } (r^j[-i])_{j \in N \diagdown \{i\}} \text{ is consensual }\}$$
$$f_i(r) = 1 \Longleftrightarrow \{S^k = \{i\} \text{ for all } k \in N \diagdown \{i\}\}$$

Proof.
Set $N^* = N \diagdown \{i\}$ and note that $f_i(r)$ is the average of the terms $f_i^{-j}(r)$ and $1 - \sum_{N^*} f_j^{-i}(r)$, all in $[0,1]$. Thus $f_i(r) = 0$ implies $\sum_{N^*} f_j^{-i}(r) = 1$. As in the proof of Theorem 1, we have

$$f_j^{-i}(r) \leq \frac{1}{1 + \sum_{k \in N^* \diagdown \{j\}} \rho^{n-3}(r_{kj}^{-i})} = \gamma_j$$

where γ_j is agent j's share in the division rule $f^{\rho^{n-3}}$ among N^*. Proposition 2 says $\sum_{N^*} \gamma_j \leq 1$, so we must have $\sum_{N^*} \gamma_j = 1$ and $f_j^{-i}(r) = \gamma_j \Leftrightarrow \rho^{n-2}(r_{ij}) = 0$ for all $j \in N^*$. By (13) the latter gives $r_{ij}^k = 0$ for all $j, k \in N^*$. By statement *iv)* in Proposition 3, $\sum_{N^*} \gamma_j = 1$ implies that the projection of r on N^* is consensual.

If $f_i(r) = 1$, each term $f_i^{-j}(r) = 1$, which by equation (10) means $\rho^{n-2}(r_{ji}) = \rho^{n-3}(r_{ki}^{-j}) = 0$ for all $k, j \neq i$. Now property (13) gives $r_{ji}^k = 0$ and the desired conclusion.■

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6 Appendix

6.1 Auxiliary results

Lemma 4
For any distinct i, j in N, pick a positive number y_{ij}, and assume $y_{ij} \cdot y_{ji} \geq 1$ for all i, j. Then

- $\sum_N \frac{1}{1+\sum_{N\setminus\{i\}} y_{ji}} \leq 1$, *and equality holds if and only if* $y_{ij} \cdot y_{ji} = 1$ *and* $y_{ij} \cdot y_{jk} \cdot y_{ki} = 1$ *for all* i, j, k *distinct;*
- *for any* $a \in \mathbb{R}^N_{++}$, $\sum_N \frac{1}{1+a_i+\sum_{N\setminus\{i\}} y_{ji}} \leq 1 - \frac{1}{1+\sum_N \frac{1}{a_i}}$, *and equality holds if and only if* $y_{ij} = \frac{a_j}{a_i}$ *for all* i, j.

Proof. We start with the second statement. Set $\varphi(y) = \sum_N \frac{1}{1+a_i+\sum_{N\setminus\{i\}} y_{ji}}$, a smooth function of y over $\mathbb{R}^{n(n-1)}_{++}$.

Our first step is to show that φ reaches its maximum in $\mathbb{R}^{n(n-1)}_{++}$. We consider the following subset A of $[\mathbb{R}_+ \cup \{+\infty\}]^{n(n-1)}$: A contains y iff for all distinct i, j, either y_{ij}, y_{ji} are both in $\mathbb{R}_{++}$ and $y_{ij} \cdot y_{ji} = 1$, or $\{y_{ij}, y_{ji}\} = \{0, \infty\}$. We extend in the obvious way the definition of φ to A, and note that the extended function is continuous on A. We let the reader check that φ reaches its maximum on A. In the summation defining $\varphi(y)$, set $z = y_{12}$, $\frac{1}{z} = y_{21}$, and note that the variable z affects the following two terms

$$\frac{1}{1+a_2+\sum_{N\setminus\{1,2\}} y_{j2} + z} + \frac{1}{1+a_1+\sum_{N\setminus\{1,2\}} y_{j1} + \frac{1}{z}}$$

Observe that for all $b, b' > 0$, the function of the variable $z, 0 \leq z \leq \infty$

$$\frac{1}{1+b+z} + \frac{1}{1+b'+\frac{1}{z}}$$

reaches its maximum uniquely at $z = \frac{b}{b'}$ (it is strictly increasing at 0, and strictly decreasing at ∞). Thus $0 < y_{12}, y_{21} < \infty$. As the choice of the pair $1, 2$ was arbitrary, we conclude that φ reaches its maximum at some $y \in \mathbb{R}^{n(n-1)}_{++}$. Moreover we have

$$y_{21} = \frac{a_1 + \sum_{j\geq 3} y_{j1}}{a_2 + \sum_{j\geq 3} y_{j2}} \Leftrightarrow y_{21}(a_2 + \sum_{j\geq 3} y_{j2}) = a_1 + \sum_{j\geq 3} y_{j1}$$

Adding $1 + y_{21}$ on both sides of this equation and taking $y_{21} \cdot y_{12} = 1$ into account, we get

$$y_{21} = \frac{1+a_1+s_1}{1+a_2+s_2} \text{ where } s_i = \sum_{N\setminus\{i\}} y_{ji} \tag{14}$$

The above equation holds for all i, j. Now compute

$$\sum_N \frac{1}{1+a_i+s_i} = \frac{1}{1+a_1+s_1} + \sum_{N\setminus\{1\}} \frac{y_{j1}}{1+a_1+s_1} = \frac{1+s_1}{1+a_1+s_1} = 1 - \frac{a_1}{1+a_1+s_1} \tag{15}$$

Again, the choice of 1 was arbitrary, therefore for all i, j

$$\frac{a_i}{1+a_i+s_i} = \frac{a_j}{1+a_j+s_j} \Rightarrow y_{ij} = \frac{a_j}{a_i}$$

and φ reaches its maximum at a single y. Moreover

$$1 + a_i + s_i = a_i(1 + \sum_N \frac{1}{a_i}) \Rightarrow \sum_N \frac{1}{1 + a_i + s_i} = \frac{\sum_N \frac{1}{a_i}}{1 + \sum_N \frac{1}{a_i}}$$

concluding the proof of the second statement.

For the first statement, we show similarly that $\psi(y) = \sum_N \frac{1}{1+\sum_{N\setminus\{i\}} y_{ji}}$ reaches its maximum at some interior points of the positive orthant, and at those points, $y_{ij} \cdot y_{ji} = 1$ for all i, j. Taking $a = 0$, the same computations as above, up to equations (14) (15), give $y_{ji} = \frac{1+s_i}{1+s_j}$ for all i, j, and $\sum_N \frac{1}{1+s_i} = 1$, as was to be proved. Note that, unlike φ, ψ reaches its maximum on an entire manifold of vectors y.■

Lemma 5.

For any distinct i, j in N, pick $y_{ij} \in [0, \infty]$, and assume $y_{ij} \cdot y_{ji} \geq 1$ for all i, j (recall our convention $0 \cdot \infty = 1$). Then $\sum_N \frac{1}{1+\sum_{N\setminus\{i\}} y_{ji}} \leq 1$, and equality holds if and only if there exists a non empty subset N_+ of N such that if we write $N_- = N \setminus N_+$, we have

$$y_{ij} = 0 \text{ if } i \in N_-, j \in N_+, \ = \infty \text{ if } i \in N_+, j \in N_-, \ 0 < y_{ij} < \infty \text{ if } i, j \in N_+ \tag{16}$$

$$y_{ij} \cdot y_{ji} = 1 \text{ and } y_{ij} \cdot y_{jk} \cdot y_{ki} = 1 \text{ for all } i, j, k \in N_+$$

Proof.

Define $N_- = \{i \in N | y_{ij} = 0 \text{ for some } j \in N\}$, $N_+ = N \setminus N_-$, and for all i, $\delta_i = \frac{1}{1+\sum_{N\setminus\{i\}} y_{ji}}$. If $i \in N_-$ and $y_{ij} = 0$, our assumption $y_{ij} \cdot y_{ji} \geq 1$ implies $y_{ji} = \infty$, hence $\delta_i = 0$. Now pick $i \in N_+$ and observe

$$\delta_i = \frac{1}{1 + \sum_{N_-} y_{ki} + \sum_{N_+ \setminus \{i\}} y_{ji}} \leq \frac{1}{1 + \sum_{N_+ \setminus \{i\}} y_{ji}} = \beta_i \tag{17}$$

The first statement of Lemma 4 applied to N_+ shows $\sum_{N_+} \beta_i \leq 1$ implying $\sum_{N_+} \delta_i \leq 1$.

It remains to check that if $\sum_N \delta_i = 1$, then y satisfies the system (16) (the converse statement following from Lemma 4). Define N_-, N_+ as above: we have $\delta_i = 0$ for $i \in N_-$ thus $\sum_{N_+} \delta_i = 1$. Combined with inequalities (17) and $\sum_{N_+} \beta_i \leq 1$, this implies $\sum_{N_+} \beta_i = 1$ and $\sum_{N_-} y_{ki} = 0$ for all $i \in N_+$, hence the first statement in (16), and the second as well because $y_{ij} \cdot y_{ji} \geq 1$. If $|N_+| = 1$, we are done, so we assume $|N_+| \geq 2$. For $i, j \in N_+$, $y_{ji} = 0$ is excluded by definition of N_+ so $y_{ji} > 0$. Assume next $y_{ji} = \infty$ for some $i, j \in N_+$which implies $\beta_i = 0$. For all $k \in N_+ \setminus \{i\}$ we have $y_{ik} > 0$ therefore

$$\beta_k = \frac{1}{1 + \sum_{N_+ \setminus \{k\}} y_{jk}} < \frac{1}{1 + \sum_{(N_+ \setminus \{i\}) \setminus \{k\}} y_{jk}} = \beta'_k$$

Now Lemma 4 gives $\sum_{N_+ \setminus \{i\}} \beta'_k \leq 1$, contradicting $\sum_{N_+ \setminus \{i\}} \beta_k = 1$. We have shown $0 < y_{ji} < \infty$ if $i, j \in N_+$. Now we apply Lemma 4 to $\sum_{N_+} \beta_i = 1$ to derive the rest of properties (16).■

6.2 A characterization of the rules f^ρ

The objective of this section is to show that the separability property described at the end of section 3 is characteristic of the rules f^ρ.

Observe that for any division rule in Definition 2 (every report awards positive shares to everyone else), agent i's payoff depends only on the reported ratios that concern him directly (r_{ij}^k for all $j,k \in N\diagdown\{i\}$). Indeed the vector $(r_{ij}^k)_{j\in N\diagdown\{i,k\}}$ entirely characterizes agent k's report, given the definition of $\mathcal{R}[N \setminus \{k\}]$. We now impose some properties on the functional dependence between agent i's payoff and the ratios r_{ij}^k for $j,k \in N\diagdown\{i\}$.

Definition 6.
*A division rule f is **non-decreasing** if $f_i(r) \geq f_i(\hat{r})$, for all i and all evaluation profiles $r,\hat{r}$ such that $r_{ij}^k \geq \hat{r}_{ij}^k$ for each $j,k \in N\diagdown\{i\}$.*

Definition 7.
*A division rule f **generates a separable ordering of the payoffs** if for all $i \in N$, and all evaluation profiles $r,\hat{r},s,\hat{s}$, for which there exists $j \in N \setminus \{i\}$ such that*

1. *$r_{ik}^l = \hat{r}_{ik}^l$ and $s_{ik}^l = \hat{s}_{ik}^l$, for all $k \neq i,j$, and all $l \neq i,k$,*
2. *$r_{ij}^l = s_{ij}^l$ and $\hat{r}_{ij}^l = \hat{s}_{ij}^l$, for all $l \neq i,j$,*

we have $f_i(r) \geq f_i(\hat{r}) \Leftrightarrow f_i(s) \geq f_i(\hat{s})$.

We interpret the separability property as follows. Condition 1 means that all the agents other than i report the same ratios in r and $\hat{r}$ (resp. s and $\hat{s}$) when comparing agent i to any other agent different from j. Under these premises, Separability says that the only information relevant to determine whether $f_i(r)$ is larger than $f_i(\hat{r})$ (resp. $f_i(s)$ is larger than $f_i(\hat{s})$) is the agents' reports concerning the pair i,j: as r and s (resp. $\hat{r}$ and $\hat{s}$) coincide as far as ij-ratios are concerned (condition 2), $f_i(r)$ is larger/smaller than $f_i(\hat{r})$ if and only if $f_i(s)$ is larger/smaller than $f_i(\hat{s})$.

Theorem 2.
Fix N, such that $|N| = n \geq 4$. A division rule f is anonymous, consensual, continuous, impartial, feasible, non-decreasing, and generates a separable ordering of the payoffs if and only if

$$f_i^\rho(r) = \frac{1}{1 + \sum_{j\in N\diagdown\{i\}} \rho(r_{ji})}$$

for some $(n-2)$-aggregator ρ that satisfies (5). Recall the notation $\rho(r_{ji}) = \rho(r_{ji}^k; k \in N\diagdown\{i,j\})$.

Proof.
If. We only need to show that f^ρ generates a separable ordering of the payoffs. The other properties have either already been discussed in section 3, or are straightforward. Consider some evaluation profiles $r,\hat{r},s,\hat{s}$ and a pair of agents i,j, as in Definition 7. Condition 1 implies $\{f_i^\rho(r) \geq f_i^\rho(\hat{r}) \Leftrightarrow \rho(r_{ji}) \leq \rho(\hat{r}_{ji})\}$

and $\{f_i^\rho(s) \geq f_i^\rho(\hat{s}) \Leftrightarrow \rho(s_{ji}) \leq \rho(\hat{s}_{ji})\}$. Condition 2 implies $\rho(r_{ji}) = \rho(s_{ji})$ and $\rho(\hat{r}_{ji}) = \rho(\hat{s}_{ji})$.

Only if. For each pair i, j in N and each $x \in \mathbb{R}_{++}^{N\setminus\{i,j\}}$, let $s(x, ij)$ be the following evaluation profile:

$$s_{kk'}^i(x, ij) = 1 \text{ for all } k, k' \neq i$$

$$s_{ij}^k(x, ij) = x_k \text{ for all } k \neq i, j$$

$$s_{ij'}^k(x, ij) = 1 \text{ for all } j' \neq j \text{ and } k \neq i, j'$$

The rest of the profile is determined by the consistency conditions (1). Define next a real valued function g_{ij} as follows:

$$g_{ij}(x) = f_i(s(x, ij)) \text{ for all } x \in \mathbb{R}_{++}^{N\setminus\{i,j\}}$$

The functions g_{ij} are all identical because f is anonymous. Let g be this common function. Anonymity implies that g is symmetric. In addition, g inherits from f the properties of non-decreasingness and continuity. For any $a, a > 0$, let $c(a) \in \mathbb{R}_{++}^{N\setminus\{i,j\}}$ be the vector $c_k(a) = a$ for all $k \in N \setminus \{i, j\}$. For all agents i, j, it is easy to construct a consensual evaluation profile $\hat{s}(c(a), ij)$ such that $\hat{s}^k(c(a), ij) = s^k(c(a), ij)$, for all $k \neq i$. Impartiality implies

$$g(c(a)) = f_i(\hat{s}(c(a), ij)) = \frac{1}{(n-1) + 1/a} \tag{18}$$

(see Lemma 1). Because g is non-decreasing, $g(x) \leq g(c(\max_{N\setminus\{i,j\}} x_k))$, hence equation (18) implies $g(x) \leq \frac{1}{n-1}$.

Define now ρ on $\mathbb{R}_{++}^{n-2}$ as follows:

$$\rho(x) = \frac{1}{g(\frac{1}{x})} - (n-1)$$

The range of ρ is $\mathbb{R}_{++}$ because that of g is $]0, \frac{1}{n-1}[$. Next ρ inherits from g the properties of continuity, non-decreasingness and symmetry. Finally $\rho(c(a)) = a$ for $a > 0$ ((18)). Thus ρ is an $(n-2)$-aggregator (Definition 4).

Let r and $\hat{r}$ be two evaluation profiles and let i be an agent. We claim

$$\{\rho(r_{ji}) = \rho(\hat{r}_{ji}) \text{for all } j \in N \diagdown \{i\}\} \Rightarrow f_i(r) = f_i(\hat{r})$$

We sketch the proof of this claim for $i = 1$ and $n = 4$. It is not difficult to extend the argument to any number of agents, as in Gorman (1968, Lemma 1). Impartiality and the consistency of the other agents' reports implies that $f_1(r)$ depends only on $(r_{21}^3, r_{21}^4, r_{13}^2, r_{13}^4, r_{14}^2, r_{14}^3)$. If $\rho(r_{21}^3, r_{21}^4) = \rho(\hat{r}_{21}^3, \hat{r}_{21}^4)$, then $g(r_{12}^3, r_{12}^4) = g(\hat{r}_{12}^3, \hat{r}_{12}^4)$, i.e. $f_1(s((r_{12}^3, r_{12}^4), 12)) = f_1(s((\hat{r}_{12}^3, \hat{r}_{12}^4), 12)$. Separability implies $f_1(r) = f_1(\tilde{r})$, where $\tilde{r}$ is such that

$$(\tilde{r}_{12}^3, \tilde{r}_{12}^4) = (\hat{r}_{12}^3, \hat{r}_{12}^4), (\tilde{r}_{13}^2, \tilde{r}_{13}^4) = (r_{13}^2, r_{13}^4), (\tilde{r}_{14}^2, \tilde{r}_{14}^3) = (r_{14}^2, r_{14}^3)$$

Similarly $\rho(r^2_{31}, r^4_{31}) = \rho(\hat{r}^2_{31}, \hat{r}^4_{31})$ implies $f_1(s((r^2_{13}, r^4_{13}), 13)) = f_1(s((\hat{r}^2_{13}, \hat{r}^4_{13}), 13)$, and by Separability $f_1(\tilde{r}) = f_1(r')$, where

$$(r'^3_{12}, r'^4_{12}) = (\tilde{r}^3_{12}, \tilde{r}^4_{12}), (r'^2_{13}, r'^4_{13}) = (\hat{r}^2_{13}, \hat{r}^4_{13}), (r'^2_{14}, r'^3_{14}) = (\tilde{r}^2_{14}, \tilde{r}^3_{14})$$

And so on.

Now we pick $i \in N$, and r an evaluation profile, both arbitrary. Define $\hat{r}$ as follows, for all $j, k \in N \diagdown \{i\}$:

$$\hat{r}^k_{ji} = \rho(r_{ji});\ \hat{r}^i_{jk} = \frac{\hat{r}^k_{ji}}{\hat{r}^j_{ki}}$$

Clearly $\rho(r_{ji}) = \rho(\hat{r}_{ji})$ for $j \in N \diagdown \{i\}$, so the claim gives $f_i(r) = f_i(\hat{r})$. On the other hand, $\hat{r}$ is consensual (Lemma 1) hence $f_i(\hat{r}) = \frac{1}{1+\sum_{j\in N\diagdown\{i\}} \rho(r_{ji})} = f^\rho_i(r)$. Finally, the feasibility of f implies that ρ satisfies (5) (iii) in Proposition 2). ■